

# MUTATION OF REPRESENTATIONS AND NEARLY MORITA EQUIVALENCE

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**ABSTRACT.** In [4] it was proved, based on [5], that the Jacobian algebra of two quivers with potential related by a QP-mutation are nearly Morita equivalent. They proved, using Axiom of Choice, that the natural functor  $\mu_k$  is an equivalence by showing that  $\mu_k$  is full, faithful and dense. In this note we provide a quasi-inverse  $\mu_k^-$  to  $\mu_k$  without Axiom of Choice.

## 1. INTRODUCTION

Let  $Q$  be an acyclic quiver,  $K$  a field and  $k \in Q_0$  a sink of  $Q$ . In [3], Bernstein-Gelfand-Ponomarev introduced a pair of adjoint functors  $(F_k^-, F_k^+)$  called now BGP reflection functors or BGP-functors. We have

$$(1.1) \quad K\langle Q \rangle\text{-mod} \xrightleftharpoons[F_k^-]{F_k^+} K\langle Q' \rangle\text{-mod} ,$$

where  $Q'$  is obtained from  $Q$  by changing the direction of all arrows incident to  $k$ . Here,  $K\langle Q \rangle$  and  $K\langle Q' \rangle$  are the path algebras of  $Q$  and  $Q'$  respectively. It was noted in [1] and [7] that these BGP-functors induce mutually quasi-inverse functors between the quotient categories  $K\langle Q \rangle\text{-mod}/[\text{add } S_k]$  and  $K\langle Q' \rangle\text{-mod}/[\text{add } S'_k]$  where  $[\text{add } S_k]$  is the ideal of morphisms that factorize through direct sums of  $S_k$ , the simple  $K\langle Q \rangle$ -module at  $k$ . Following Ringel, [9], we say in this case that  $K\langle Q \rangle$  and  $K\langle Q' \rangle$  are *nearly Morita equivalent*.

More generally one can consider, for a 2-acyclic quiver and any  $k \in Q_0$ , the quiver mutation  $\mu_k(Q)$ , see for example [8]. However, in this case we may not expect to relate  $K\langle Q \rangle\text{-mod}$  and  $K\langle \mu_k(Q) \rangle\text{-mod}$  in a meaningful way.

In [5], Derksen, Weyman and Zelevinsky defined mutations of quivers with potential and also they defined mutations of the representations of a quiver with potential. In [5, Theorem 10.13] it was proved that the mutation of representations is an involution up to right-equivalence.

In [4, Section 7] a natural functor  $\mu_k$  is constructed. This functor  $\mu_k$  is defined from  $\mathcal{P}(Q, S)\text{-mod}/[\text{add } S_k]$  to  $\mathcal{P}(\mu_k(Q, S))\text{-mod}/[\text{add } S'_k]$ , where  $\mathcal{P}(Q, S)$  denotes the Jacobian algebra of a quiver with potential  $(Q, S)$ . This functor is based on the notion of mutation of representation introduced in [5]. By making use of [5, Theorem 10.13] it was proved, [4, Theorem 7.1], that  $\mu_k$  is full, faithful and dense. It is well known that this implies, by the Axiom of Choice, that the functor is an equivalence (see, for example [2, Appendix A, 2.5]).

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Let us write from now on  $\mu_k^+ := \mu_k$ . In this note we produce explicitly a quasi-inverse  $\mu_k^-$  of  $\mu_k^+$ , which is, in fact, quite similar to  $\mu_k^+$ .

Since we are interested in the factor category  $\mathcal{P}(Q, S)\text{-mod} / [\text{add } S_k]$  we avoid to work with decorated representations. To give the quasi-inverse  $\mu_k^-$  explicitly we proceed as follows. First at all, following [5], for the convenience of the reader in Section 2 and Section 3 we recall some background on mutation of quivers with potential and the mutation of their representations. The rest of this note is devoted to define  $\mu_k^+$ ,  $\mu_k^-$ , and show that they are quasi-inverses, see Theorem 5.1.

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## 2. BACKGROUND ON QUIVERS WITH POTENTIAL

A *quiver*  $Q = (Q_0, Q_1, t, h)$  consists of a finite set of vertices  $Q_0$ , a finite set of arrows  $Q_1$  and two maps  $t, h : Q_1 \rightarrow Q_0$  (head, and tail). For each  $a \in Q_1$  we write  $a : t(a) \rightarrow h(a)$ . Given a algebraically closed field  $K$  we will denote by  $R = \times_{i \in Q_0} K$  to the *vertex space* and by  $A = \times_{a \in Q_1} K$  to the *arrow space*. We have  $R$  is a semisimple  $K$ -algebra with the usual addition and multiplication defined coordinate-wise and  $A$  is an  $R$ -bimodule with following structure:

$$\begin{aligned} (x_l)_{l \in Q_0} \cdot ((y_a)_{a \in Q_1}) &= (x_{h(a)} y_a)_{a \in Q_1} \\ ((y_a)_{a \in Q_1}) \cdot (x_l)_{l \in Q_0} &= (y_a x_{t(a)})_{a \in Q_1}. \end{aligned}$$

For  $l > 0$ , let  $A^l = A \otimes_R \cdots \otimes_R A$  be the  $l$ -fold tensor product over  $R$  of  $A$  with itself as  $R$ -bimodule. The *path algebra* of  $Q$  is defined as the tensor algebra  $K\langle Q \rangle = \bigoplus_{l \geq 0} A^l$  and the *complete path algebra* is defined as the complete tensor algebra  $K\langle\langle Q \rangle\rangle = \prod_{l \geq 0} A^l$ .

We say that a sequence of arrows  $\alpha = a_l a_{l-1} \cdots a_2 a_1$ , is a *path* of  $Q$  if  $t(a_{k+1}) = h(a_k)$ , for  $k = 1, \dots, l-1$ , in this case, we define the *length* of  $\alpha$  as  $l$ . We say that  $\alpha$  is a *cycle* if  $h(a_l) = t(a_1)$ . Then we can think the elements of  $K\langle Q \rangle$  as  $K$ -linear combinations of paths and the elements of  $K\langle\langle Q \rangle\rangle$  as possibly infinite  $K$ -linear combinations of paths.

Let us recall some important fact about the complete path algebra  $K\langle\langle Q \rangle\rangle$ . Let  $\mathfrak{M} = \prod_{l \geq 1} A^l$  be the two-sided ideal of  $K\langle\langle Q \rangle\rangle$  generated by arrows of  $Q$ . Then  $K\langle\langle Q \rangle\rangle$  can be viewed as a *topological  $K$ -algebra* with the powers of  $\mathfrak{M}$  as a basic system of open neighborhoods of 0. This topology is known as  *$\mathfrak{M}$ -adic topology*. Now, giving  $I \subseteq K\langle\langle Q \rangle\rangle$  we can calculate the closure of  $I$  as  $\bar{I} = \bigcap_{l \geq 0} (I + \mathfrak{M}^l)$ .

We say, by a slight abuse of language, that a  $K$ -algebras homomorphism  $\phi : K\langle\langle Q \rangle\rangle \rightarrow K\langle\langle Q' \rangle\rangle$  is a *homomorphism of  $R$ -algebras* if  $\phi(r) = r$  for each  $r \in R$ . If this is the case,  $\phi$  is continuous as morphism of topological algebras (see [5, Section 2]).

A *finite-dimensional representation* of  $Q$  over  $K$  is a pair  $((M_i)_{i \in Q_0}, (M_a)_{a \in Q_1})$  where  $M_i$  is a finite-dimensional vector space over  $K$  for each  $i \in Q_0$  and  $M_a : M_{t(a)} \rightarrow M_{h(a)}$  is a  $K$ -linear map. Here the word *representation* means finite dimensional representation. We say that  $M$  is a *nilpotent* representation if there is an  $n > 0$  such that for every path  $a_n a_{n-1} \cdots a_1$  of length  $n$  in  $Q$  we have  $M_{a_n} M_{a_{n-1}} \cdots M_{a_1} = 0$ .

We denote by  $\text{nil}_K(Q)$  the category of nilpotent representations of  $Q$ , and by  $K\langle\langle Q \rangle\rangle\text{-mod}$  the category of finite-dimensional left  $K\langle\langle Q \rangle\rangle$ -modules. It is well

known that the category of representations of  $Q$  and the category of  $K\langle Q \rangle$ -modules are equivalent. In [5, Section 10] it was observed that  $\text{nil}_K(Q)$  and  $K\langle\langle Q \rangle\rangle\text{-mod}$  are equivalent.

**2.1. Quivers with potential and their mutations.** In this preliminary section, for the convenience of the reader, we shall recall some basic definitions and facts from [5]. Let  $Q$  be a quiver. We say that  $S \in K\langle\langle Q \rangle\rangle$  is a *potential* for  $Q$  if  $S$  is a, possibly infinite,  $K$ -linear combination of cycles in  $Q$ . Given two potentials  $S$  and  $W$  we say that they are *cyclically equivalent* and write  $S \sim_{\text{cyc}} W$ , if  $S - W$  is in the closure of the sub-vector space of  $K\langle\langle Q \rangle\rangle$  generated by all elements of the form  $a_1 a_2 \cdots a_{n-1} a_n - a_2 \cdots a_{n-1} a_n a_1$ , with  $a_1 a_2 \cdots a_{n-1} a_n$  a cycle on  $Q$ .

**Definition 2.1.** We say  $(Q, S)$  is a *quiver with potential* (QP) if  $Q$  does not have loops,  $S$  is a potential for  $Q$  and if any two different cycles appearing with non-zero coefficient in  $S$  are not cyclically equivalent.

Given  $a \in Q_1$  and a cycle  $a_n a_{n-1} \cdots a_1$  in  $Q$ , define the *cyclic derivative* of  $a_n a_{n-1} \cdots a_1$  with respect to  $a$  as follows:

$$\partial_a(a_n a_{n-1} \cdots a_1) = \sum_{k=1}^n \delta_{a, a_k} a_{k-1} a_{k-2} \cdots a_1 a_n a_{n-1} \cdots a_{k+2} a_{k+1}.$$

We extend this definition by  $K$ -linearity and continuity to all potentials for  $Q$ .

**Definition 2.2.** Let  $(Q, S)$  be a quiver with potential. We define the *Jacobian ideal*  $\mathcal{J}(Q, S)$  as the closure of the ideal on  $K\langle\langle Q \rangle\rangle$  generated by all cyclic derivatives  $\partial_a(S)$  with  $a \in Q_1$ . The quotient  $K\langle\langle Q \rangle\rangle / \mathcal{J}(Q, S)$  is called the *Jacobian algebra* of  $(Q, S)$  and is denoted as  $\mathcal{P}(Q, S)$ .

A quiver with potential  $(Q, S)$  is *trivial* if the  $R$ - $R$ -bimodule on  $k\langle\langle Q \rangle\rangle$  generated by  $\partial_a(S)$  is  $A$  with  $a \in Q_1$ . We call  $(Q, S)$  *reduced* if  $S$  does not have cycles of length 2.

**Definition 2.3.** Let  $(Q, S)$  and  $(Q', S')$  be quivers with potential. We say  $\phi : K\langle\langle Q \rangle\rangle \rightarrow K\langle\langle Q' \rangle\rangle$  is a *right-equivalence* of QP if  $\phi$  is a  $R$ -algebra isomorphism and  $\phi(S) \sim_{\text{cyc}} S'$ .

In [5, Theorem 4.6] it was proved that given a quiver with potential  $(Q, S)$ , there are a trivial quiver with potential  $(Q_{\text{triv}}, S_{\text{triv}})$ , a reduced quiver with potential  $(Q_{\text{red}}, S_{\text{red}})$  and a right-equivalence  $\phi$  such that  $\phi(S) \sim_{\text{cyc}} S_{\text{triv}} + S_{\text{red}}$ . This result is called “the splitting Theorem”.

Let us recall the notion of quivers mutation. Let  $Q$  be a quiver and  $k \in Q_0$  a vertex, if  $Q$  does not have any cycle of length 2 (2-cycle) based at  $k$ , the mutation of  $Q$  with respect to  $k$ , denoted  $\mu_k(Q)$ , can be obtained by the next three steps.

- (1) For each pair of arrows  $a : j \rightarrow k$  and  $b : k \rightarrow i$ , add a new arrow  $[ba] : j \rightarrow i$ .
- (2) Replace each arrow  $a : j \rightarrow k$  with  $a^* : k \rightarrow j$  and  $b : k \rightarrow i$  with  $b^* : i \rightarrow k$ .
- (3) Remove any maximal disjoint collection of oriented 2-cycles.

Now we define QP-mutation, following [5]. Let  $(Q, S)$  be a quiver with potential. Suppose that  $Q$  is 2-acyclic and there are no cycles in  $S$  that begin at  $k$ . We define the *mutation* of  $(Q, S)$  with respect to  $k$  as  $\mu_k(Q, S) = (\tilde{\mu}_k(Q)_{\text{red}}, \tilde{S}_{\text{red}})$ , where

$\tilde{\mu}_k(Q)$  is obtained from  $Q$  by applying the two first steps of  $k^{\text{th}}$ -mutation  $\mu_k$  and  $\tilde{S}$  is:

$$(2.1) \quad \tilde{S} = [S] + \sum_{a,b \in Q_1: h(a)=k=t(b)} [ba]a^*b^*,$$

where  $[S]$  is obtained from  $S$  by replacing any pair of arrows  $a : j \rightarrow k$  and  $b : k \rightarrow i$  in the expansion of  $S$  with  $[ba]$ . In [5] is showed that  $\mu_k(Q, S)$  is well defined up to right equivalence.

**2.2. Mutation of representations.** Let  $(Q, S)$  be a quiver with potential. We say that  $M$  is a *representation* of  $(Q, S)$  if  $M$  is a finite-dimensional representation of  $Q$  and  $M$  satisfies the cyclic derivatives  $\partial_a(S)$  for each  $a \in Q_1$ . Now we extend the notion of right-equivalence from quivers with potential to their representations. Our definition is a bit different from [5, Definition 10.2].

**Definition 2.4.** Let  $M$  and  $M'$  be representations of  $(Q, S)$  and  $(Q', S')$ . A pair  $(\phi, \psi)$  is a *right-equivalence* from  $M$  to  $M'$  if it satisfies the following:

- (1)  $\phi : K\langle\langle Q \rangle\rangle \rightarrow K\langle\langle Q' \rangle\rangle$  is a right-equivalence between  $(Q, S)$  and  $(Q', S')$ .
- (2)  $\psi : M \rightarrow M'$  is a vector space isomorphism such that  $\psi \circ (u \cdot -) = (\phi(u) \cdot -) \circ \psi$ , for every  $u \in K\langle\langle Q \rangle\rangle$ .

*Remark 2.5.* Denote  $\Lambda = K\langle\langle Q \rangle\rangle$ . Note that if we define a  $\Lambda$ -module  $M'^\phi$  as  $M'$  like vector space with the structure  $u \cdot m' = \phi(u) \cdot m'$  the second condition of Definition 2.4 can be restated by saying that  $\psi : M \rightarrow M'^\phi$  is a  $\Lambda$ -module isomorphism.

**Definition 2.6.** Let

$$\phi : K\langle\langle Q_{\text{triv}} \oplus Q_{\text{red}} \rangle\rangle \rightarrow K\langle\langle Q \rangle\rangle$$

be a right-equivalence. We define the *reduced part* of  $M$ , denoted  $M_{\text{red}}$ , as  $M^{\phi_r}$  (see Remark 2.5), where  $\phi_r$  is the restriction of  $\phi$  to  $K\langle\langle Q_{\text{red}} \rangle\rangle$ .

Let  $(Q, S)$  be a quiver with potential and suppose that  $Q$  is 2-acyclic. Mutation of representations of  $(Q, S)$  was introduced in [5]. Let  $M$  be a representation of  $(Q, S)$  and  $k$  a vertex in  $Q$ . Set  $\{a_1, a_2, \dots, a_s\} = \{a \in Q_1 \mid h(a) = k\}$  with  $a_i \neq a_j$  for  $i \neq j$  and  $\{b_1, b_2, \dots, b_t\} = \{b \in Q_1 \mid t(b) = k\}$  with  $b_i \neq b_j$  for  $i \neq j$ . Define

$$(2.2) \quad M_{\text{in}}(k) = \bigoplus_{p=1}^s M_{t(a_p)}, \quad M_{\text{out}}(k) = \bigoplus_{q=1}^t M_{h(b_q)}$$

We get natural linear maps  $\alpha_{k,M} : M_{\text{in}}(k) \rightarrow M_k$  and  $\beta_{k,M} : M_k \rightarrow M_{\text{out}}(k)$ , in matrix form

$$(2.3) \quad \alpha_{k,M} = (a_1 \dots a_s), \quad \beta_{k,M} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_t \end{pmatrix}$$

Let  $c_1 \dots c_n$  be a cycle in  $Q$ , we define

$$(2.4) \quad \partial_{b_q a_p}(c_1 \dots c_n) = \sum_{j=1}^n \delta_{b_q a_p, c_j c_{j+1}} c_{j+2} \dots c_n c_1 \dots c_{j-1},$$

for  $p = 1 \dots s$  and  $q = 1 \dots t$ . We extend this definition by  $K$ -linearity and continuity to all potentials for  $Q$ .

Define the linear map  $\gamma_{k,M} : M_{\text{out}}(k) \longrightarrow M_{\text{in}}(k)$  in matrix form

$$(2.5) \quad (\gamma_{k,M})_{p,q} = \partial_{b_q a_p}(S) : M_{h(b_q)} \longrightarrow M_{t(a_p)}$$

It is useful to keep in mind the following *local triangle associated to  $M$  at  $k$* , which summarizes the data that we got so far:

$$(2.6) \quad \begin{array}{ccc} & M_k & \\ \alpha_{k,M} \nearrow & & \searrow \beta_{k,M} \\ M_{\text{in}}(k) & \xleftarrow{\gamma_{k,M}} & M_{\text{out}}(k) \end{array}$$

We write  $(\tilde{Q}, \tilde{S}) = \tilde{\mu}_k(Q, S)$ . In what follows we define the *pre-mutation*  $\tilde{\mu}_k(M)$  as a representation of  $(\tilde{Q}, \tilde{S})$ , for short  $\overline{M} := \tilde{\mu}_k(M)$ . As vector space,  $\overline{M}_i = M_i$ , if  $i \neq k$ , and if  $i = k$  we set

$$(2.7) \quad \overline{M}_k = \frac{\ker(\gamma_{k,M})}{\text{im}(\beta_{k,M})} \oplus \text{im}(\gamma_{k,M}) \oplus \frac{\ker(\alpha_{k,M})}{\text{im}(\gamma_{k,M})}.$$

For the action of the arrows we note that  $\overline{M}_{\text{in}}(k) = M_{\text{out}}(k)$  and  $\overline{M}_{\text{out}}(k) = M_{\text{in}}(k)$ , now define

$$(2.8) \quad \alpha_{k,\overline{M}} = (b_1^*, \dots, b_t^*), \quad \beta_{k,\overline{M}} = \begin{pmatrix} a_1^* \\ a_2^* \\ \vdots \\ a_s^* \end{pmatrix}.$$

We choose a retraction and a section

$$(2.9) \quad \rho_M : M_{\text{out}}(k) \longrightarrow \ker(\gamma_{k,M}), \quad \sigma_M : \ker(\alpha_{k,M}) / \text{im}(\gamma_{k,M}) \longrightarrow \ker(\alpha_{k,M})$$

In other words, we have  $\rho_M \iota = \text{id}_{\ker(\gamma_{k,M})}$  and  $\pi \sigma_M = \text{id}_{\ker(\alpha_{k,M}) / \text{im}(\gamma_{k,M})}$ , though  $\iota$  and  $\pi$  are the natural inclusion and projection respectively. We are ready to define the action of  $\tilde{Q}$  in  $\overline{M}$

$$(2.10) \quad \alpha_{k,\overline{M}} = \begin{pmatrix} -\pi \rho_M \\ -\gamma_{k,M} \\ 0 \end{pmatrix}, \quad \beta_{k,\overline{M}} = (0, \iota, \iota \sigma_M).$$

In [5] the *mutation* in direction  $k$  is defined as  $\tilde{\mu}_k(M)_{\text{red}} = \overline{M}_{\text{red}}$ , see Definition 2.6. There, the authors proved that  $\overline{M}$  is actually a representation of  $\tilde{\mu}_k(Q, S)$  and that it does not depend of the splitting data (2.9), up to isomorphism, see [5, Proposition 10.9]. Another fact proved in the same work is that the class of right-equivalence of  $\tilde{\mu}_k(M)_{\text{red}}$  is determined by the class of right-equivalence of  $M$ , see [5, Proposition 10.10].

## 3. INVOLUTIVITY OF MUTATION

Let  $(Q, S)$  be a reduced QP with no 2-cycles at  $k$ , and  $M$  be a representation of  $(Q, S)$ . Denote by  $\overline{M}$  the representation  $\tilde{\mu}_k(\tilde{\mu}_k(M))$  of  $(\tilde{Q}, \tilde{S}) = \tilde{\mu}_k(\tilde{\mu}_k(Q, S))$ . From [5, Theorem 5.7], [5, Proposition 10.11] and Remark 2.5 we can see that  $\mu_k^2(M) = \overline{M}_{\text{red}} = \overline{M}^\varphi$ , where  $\varphi$  is defined as follows

$$(3.1) \quad \varphi : b_q \longmapsto -b_q, \text{ for } q = 1, \dots, t \text{ and } \varphi \text{ fix the rest of the arrows in } \tilde{Q}.$$

Where  $\varphi$  is an automorphism of  $K\langle\langle\tilde{Q}\rangle\rangle$  and we are identifying the arrows  $b_q$  on  $Q$  with the arrows  $b_q^{**}$  on  $\tilde{Q}$  for  $q = 1, \dots, t$ ; see the proof of [5, Theorem 5.7].

In [5, Theorem 10.13] was proved that  $M$  is right-equivalent to  $\mu_k^2(M)$ . In fact, we can deduce from the proof of [5, Theorem 10.13] a slightly sharper statement.

**Lemma 3.1.** *Let  $(Q, S)$  be a quiver with potential and  $M$  be a representation of  $(Q, P)$  such that  $M$  does not have direct summands isomorphic to  $S_k$ . If  $\varphi$  is as in (3.1), then  $M$  is isomorphic to  $\overline{M}^\varphi$ .*

4. DEFINITION OF THE FUNCTOR  $\mu_k^+$ 

Giving a reduced quiver with potential  $(Q, S)$ , let  $\mathcal{P}(Q, S)$  be the Jacobian algebra of  $(Q, S)$ . We write  $\Lambda = \mathcal{P}(Q, S)$  and  $\Lambda' = \mathcal{P}(\mu_k(Q, S))$ .

We fix a vertex  $k \in Q_0$ . If  $f : M \rightarrow N$  be a morphism of representations of  $(Q, S)$ , we say that  $f$  is *confined to  $k$*  if  $f(m) = 0$ ,  $\forall m \in M_{\hat{k}}$ , with  $M_{\hat{k}} = \bigoplus_{i \neq k} M_i$ . The set of all morphisms confined to  $k$  is denoted by  $\text{Hom}_{\Lambda\text{-mod}}^k(M, N)$ . From [6, Section 6] we have that

$$\text{Hom}_{\Lambda\text{-mod}/[\text{add } S_k]}(M, N) = \text{Hom}_{\Lambda\text{-mod}}(M, N) / \text{Hom}_{\Lambda\text{-mod}}^k(M, N).$$

In [4, Theorem 7.1] and [6, Proposition 6.2] were proved that there is an equivalence between  $\Lambda\text{-mod}/[\text{add } S_k]$  and  $\Lambda'\text{-mod}/[\text{add } S'_k]$ . Their proofs use the Axiom of Choice, indeed their arguments consist by showing that a functor  $F$  is full, faithful and dense, but it is well known that  $F$  is an equivalence thanks to the Axiom of Choice. In this note we provide an explicit quasi-inverse of the equivalence defined in [4, Theorem 7.1].

From [4] the functor  $\mu_k^+ : \Lambda\text{-mod}/[\text{add } S_k] \rightarrow \Lambda'\text{-mod}/[\text{add } S'_k]$  defined below is an equivalence. If  $M \in \Lambda\text{-mod}$ , then  $\mu_k^+(M)$  is defined as (2.7). Now, given  $f \in \text{Hom}_{\Lambda\text{-mod}}(M, N)$  we proceed as follows to define  $\mu_k(f)$ . We set  $\mu_k(f)_j := f_j$  for  $j \neq k$ . In order to define  $\mu_k^+(f)_k : \overline{M}_k \rightarrow \overline{N}_k$  we consider the following diagram obtained from (2.7)

$$(4.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \frac{\ker(\gamma_{k,M})}{\text{im}(\beta_{k,M})} & \xrightarrow[\tilde{\rho}_M]{\tilde{\gamma}_M} & \text{coker}(\beta_{k,M}) & \xrightarrow[\tilde{\gamma}_{k,M}]{\iota_M \tilde{\gamma}_{k,M}} & \ker(\alpha_{k,M}) \xrightarrow[\sigma_M]{\pi} \frac{\ker(\alpha_{k,M})}{\text{im}(\gamma_{k,M})} \longrightarrow 0 \\ & & & & \swarrow \tilde{\gamma}_{k,M} & \searrow \iota_M & \\ & & & & \text{im}(\gamma_{k,M}) & & \\ & & \swarrow j_M & & \searrow \epsilon_M & & \\ & 0 & & & & & 0 \end{array}$$

With  $\tilde{i}_M: \frac{\ker(\gamma_{k,M})}{\text{im}(\beta_{k,M})} \rightarrow \text{coker}(\beta_{k,M})$  and  $\tilde{\gamma}_{k,M}: \text{coker}(\beta_{k,M}) \rightarrow \text{im}(\gamma)$  the natural induced maps. The map  $\tilde{\rho}_M: \text{coker}(\beta_{k,M}) \rightarrow \frac{\ker(\gamma_{k,M})}{\text{im}(\beta_{k,M})}$  is induced by  $\rho_M$  (see (2.9)). The choice of  $\tilde{\rho}_M$  and  $\sigma_M$  allow to define maps  $j_M$  and  $\epsilon_M$  respectively, such that  $\text{id}_{\text{coker}(\beta_{k,M})} = i_M \tilde{\rho}_M + j_M \tilde{\gamma}_{k,M}$  and  $\text{id}_{\ker(\alpha_{k,M})} = \iota_M \epsilon_M + \sigma_M \pi$ .

We get a similar diagram for  $N$ , then we obtain the following diagram where the central square is commutative

$$(4.2) \quad \begin{array}{ccccc} \frac{\ker(\alpha_{k,M})}{\text{im}(\gamma_{k,M})} & & \frac{\ker(\alpha_{k,N})}{\text{im}(\gamma_{k,N})} & & \\ \sigma_M \searrow & & \nearrow \pi' & & \\ \oplus & \text{ker}(\alpha_{k,M}) & \xrightarrow{f_{\text{in}}} & \text{ker}(\alpha_{k,N}) & \oplus \\ \uparrow \iota_M & \uparrow \tilde{\gamma}_{k,M} & & \uparrow \tilde{\gamma}_{k,N} & \uparrow \epsilon_N \\ \text{im}(\gamma_{k,M}) & & & & \text{im}(\gamma_{k,N}) \\ \downarrow j_M & & & & \downarrow \tilde{\gamma}_{k,N} \\ \oplus & \text{coker}(\beta_{k,M}) & \xrightarrow{f_{\text{out}}} & \text{coker}(\beta_{k,N}) & \oplus \\ \nearrow \tilde{i}_M & & & & \searrow \tilde{\rho}_N \\ \frac{\ker(\gamma_{k,M})}{\text{im}(\beta_{k,M})} & & & & \frac{\ker(\gamma_{k,N})}{\text{im}(\beta_{k,N})} \end{array}$$

Here  $\overline{f_{\text{out}}}$  is the induced map by  $f_{\text{out}}: M_{\text{out}}(k) \rightarrow N_{\text{out}}(k)$ . Now we can already give the definition of  $\mu_k^+(f)_k$ :

$$(4.3) \quad \mu_k^+(f)_k = \begin{pmatrix} \tilde{\rho}_N \overline{f_{\text{out}}} \tilde{i}_M & \tilde{\rho}_N \overline{f_{\text{out}}} j_M & 0 \\ \tilde{\gamma}_{k,N} \overline{f_{\text{out}}} \tilde{i}_M & \epsilon_N f_{\text{in}} \iota_M & \epsilon_N f_{\text{in}} \sigma_M \\ 0 & \pi' f_{\text{in}} \iota_M & \pi' f_{\text{in}} \sigma_M \end{pmatrix}.$$

## 5. QUASI-INVERSE

The notion of right-equivalence of representations, see Definition 2.4, is a delicate point in this subject. One reason is that the classes of right-equivalence may have a different behavior of isomorphism classes, for example see [5][Remark 10.3]. Indeed, it is possible to find two no isomorphic representations that are right-equivalent. Since [5][Theorem 10.13] is stated in terms of the notion of right-equivalence, it is not obvious that one may say something about isomorphism classes as in the proof of [4][Theorem 7.1]. In this section we use the notion of right-equivalence for representations of a quiver with potential to define explicitly a quasi inverse for  $\mu_k^+$ .

Let  $(P, T)$  be a reduced quiver with potential with no 2-cycles at  $k \in P_0$ . We define  $\tilde{\mu}_k^-(P, T)$  as  $\tilde{\mu}_k^-(P, T) = (\tilde{\mu}_k(P), \tilde{T}^-)$ , with

$$(5.1) \quad \tilde{T}^- = [T] - \sum_{\substack{a, b \in P_1: \\ h(a)=k=t(b)}} [ba] a^* b^*$$

We define  $\mu_k^-(P, T)$  as the reduced part of  $\tilde{\mu}_k^-(P, T)$ . This notion of mutation induces a mutation of representations of  $(P, T)$ . Let  $M$  be a representation of  $(P, T)$ . We define  $\tilde{\mu}_k^-(M)$  as vector space the same form that  $\tilde{\mu}_k^+(M)$ , (see (2.7)). We need define the actions of arrows in  $\tilde{\mu}_k^-(M)$ , (see (2.8)). After choose maps as (2.9) we obtain the new version of (2.10)

$$(5.2) \quad \alpha_{k, \overline{M}}^- = \begin{pmatrix} \pi \rho_M \\ \gamma_{k, M} \\ 0 \end{pmatrix}, \quad \beta_{k, \overline{M}}^- = (0, \iota, \iota \sigma_M)$$

Then we define  $\mu_k^-(M) = \tilde{\mu}_k^-(M)_{\text{red}}$ . This is well defined up to right-equivalence.

Returning to our task, we have  $\mu_k^- : \Lambda' \text{ mod } [\text{add } S'_k] \rightarrow \Lambda \text{ mod } [\text{add } S_k]$ , the notation is the same to the last section.

**Theorem 5.1.**  $\mu_k^-$  is a quasi-inverse of  $\mu_k^+$ .

*Proof.* Let us write  $M'$  for  $\mu_k^- \mu_k^+(M)$  to relax the notation. First, we compute  $M'$  as vector space. We then describe the structure of  $M'$  as  $K\langle\langle Q \rangle\rangle$ -module.

By definition we have  $M'_i = M_i$ ,  $i \neq k$ . Now from (2.7) we can deduce that

$$(5.3) \quad M'_k = \frac{\ker(\gamma_{k, \overline{M}})}{\text{im}(\beta_{k, \overline{M}})} \oplus \text{im}(\gamma_{k, \overline{M}}) \oplus \frac{\ker(\alpha_{k, \overline{M}})}{\text{im}(\gamma_{k, \overline{M}})},$$

where  $\overline{M} = \tilde{\mu}_k^+(M)$  as representation of  $\tilde{\mu}_k^+(Q, S)$ . Applying the definitions we see

$$(5.4) \quad \begin{aligned} \ker(\alpha_{k, \overline{M}}) &= \text{im}(\beta_{k, M}), \quad \text{im}(\alpha_{k, \overline{M}}) = \frac{\ker(\gamma_{k, M})}{\text{im}(\beta_{k, M})} \oplus \text{im}(\gamma_{k, M}) \oplus \{0\}, \\ \ker(\beta_{k, \overline{M}}) &= \frac{\ker(\gamma_{k, M})}{\text{im}(\beta_{k, M})} \oplus \{0\} \oplus \{0\}, \quad \text{im}(\beta_{k, \overline{M}}) = \ker(\alpha_{k, M}), \\ \ker(\gamma_{k, \overline{M}}) &= \ker(\beta_{k, M} \alpha_{k, M}) \quad \text{im}(\gamma_{k, \overline{M}}) = \text{im}(\beta_{k, M} \alpha_{k, M}). \end{aligned}$$

By rewriting (5.3) we get

$$(5.5) \quad M'_k = \frac{\ker(\beta_{k, M} \alpha_{k, M})}{\ker(\alpha_{k, M})} \oplus \text{im}(\beta_{k, M} \alpha_{k, M}) \oplus \frac{\text{im}(\beta_{k, M})}{\text{im}(\beta_{k, M} \alpha_{k, M})}$$

Now we observe:

- $\alpha_{k, M}$  induces an isomorphism

$$\tilde{\alpha} : \frac{\ker(\beta_{k, M} \alpha_{k, M})}{\ker(\alpha_{k, M})} \longrightarrow \ker(\beta_{k, M}), \quad [x] \mapsto \alpha_{k, M}(x).$$

- $\beta_{k, M}$  induces an isomorphism

$$\tilde{\beta} : \frac{\text{im}(\alpha_{k, M})}{\ker(\beta_{k, M})} \longrightarrow \text{im}(\beta_{k, M} \alpha_{k, M}), \quad [x] \mapsto \beta_{k, M}(x).$$

- $\beta_{k, M}$  induces an isomorphism

$$\hat{\beta} : \frac{M_k}{\text{im}(\alpha_{k, M})} \longrightarrow \frac{\text{im}(\beta_{k, M})}{\text{im}(\beta_{k, M} \alpha_{k, M})}, \quad [x] \mapsto [\beta_{k, M}(x)].$$



Since  $M$  does not have direct summands isomorphic to  $S_k$  we have  $\ker(\beta_{k,M}) \subseteq \text{im}(\alpha_{k,M})$ . Combine (5.4) with the induced isomorphisms above to represent  $M'_k$  in the following way

$$(5.6) \quad M'_k = \ker(\beta_{k,M}) \oplus \frac{\text{im}(\alpha_{k,M})}{\ker(\beta_{k,M})} \oplus \frac{M_k}{\text{im}(\alpha_{k,M})}.$$

With this form for  $M'_k$  we choose linear maps as in (2.9), that is

$$\rho_{\overline{M}} : M_{\text{in}}(k) \rightarrow \ker(\beta_{k,M} \alpha_{k,M}), \quad \sigma_{\overline{M}} : \text{im}(\beta_{k,M}) / \text{im}(\beta_{k,M} \alpha_{k,M}) \rightarrow \text{im}(\beta_{k,M}).$$

Thus,  $\rho_{\overline{M}} \iota = \text{id}_{\ker(\beta_{k,M} \alpha_{k,M})}$ , and  $\pi \sigma_{\overline{M}} = \text{id}_{\text{im}(\beta_{k,M}) / \text{im}(\beta_{k,M} \alpha_{k,M})}$ . Now we can define the structure of  $M'$  as a representation. To do that, define linear maps as in (2.10):

$$(5.7) \quad \alpha_{k,M'}^- = \begin{pmatrix} \alpha_{k,M} \rho_{\overline{M}} \\ \pi \alpha_{k,M} \\ 0 \end{pmatrix}, \quad \beta_{k,M'}^- = (0, \tilde{\beta}_{k,M}, \iota \sigma_{\overline{M}} \hat{\beta}_{k,M}).$$

Note that in this case we do not need  $\varphi$ , see (3.1) and the proof of [5, Theorem 5.7].

Let  $f : M \rightarrow N$  be a morphism of representations of  $(Q, S)$ . With the form of  $\mu_k \mu_k^+(M)$  we can deduce the new version of (4.1) and (4.2). Again the center square is commutative and  $\tilde{f}_{\text{in}}$  is induced by  $f_{\text{in}}$ .

$$(5.8) \quad \begin{array}{ccccc} \frac{M_k}{\text{im}(\alpha_{k,M})} & & & & \frac{N_k}{\text{im}(\alpha_{k,N})} \\ & \searrow \sigma_{\overline{M}} \hat{\beta}_{k,M} & & & \searrow \hat{\beta}_{k,N}^{-1} \pi \\ \oplus & \text{im}(\beta_{k,M}) & \xrightarrow{f_{\text{out}}} & \text{im}(\beta_{k,N}) & \oplus \\ & \nearrow \iota_{\overline{M}} \tilde{\beta}_{k,M} & & & \nearrow \tilde{\beta}_{k,N}^{-1} \epsilon_{\overline{N}} \\ \frac{\text{im}(\alpha_{k,M})}{\ker(\beta_{k,M})} & & & & \frac{\text{im}(\alpha_{k,N})}{\ker(\beta_{k,N})} \\ & \searrow j_{\overline{M}} \tilde{\beta}_{k,M} & & & \searrow \tilde{\beta}_{k,M}^{-1} \tilde{\gamma}_{k,N} \\ \oplus & \frac{M_{\text{in}}(k)}{\ker(\alpha_{k,M})} & \xrightarrow{\tilde{f}_{\text{in}}} & \frac{N_{\text{in}}(k)}{\ker(\alpha_{k,N})} & \oplus \\ & \nearrow \tilde{\gamma}_{k,M} \tilde{\alpha}_{k,M}^{-1} & & & \nearrow \tilde{\alpha}_{k,N} \tilde{\rho}_{\overline{N}} \\ \ker(\beta_{k,M}) & & & & \ker(\beta_{k,N}) \end{array}$$

Finally we define  $\mu_k^- \mu_k^+(f)_k$  and denote it as  $f'_k$ :

$$f'_k = \begin{pmatrix} \tilde{\alpha}_{k,N} \tilde{\rho}_{\overline{N}} \tilde{f}_{\text{in}} \tilde{\iota}_{\overline{M}} \tilde{\alpha}_{k,M}^{-1} & \tilde{\alpha}_{k,N} \tilde{\rho}_{\overline{N}} \tilde{f}_{\text{in}} j_{\overline{M}} \tilde{\beta}_{k,M} & 0 \\ \tilde{\beta}_{k,N}^{-1} \tilde{\gamma}_{k,N} \tilde{f}_{\text{in}} \tilde{\iota}_{\overline{M}} \tilde{\alpha}_{k,M}^{-1} & \tilde{\beta}_{k,N}^{-1} \epsilon_{\overline{N}} f_{\text{out}} \iota_{\overline{M}} \tilde{\beta}_{k,M} & \tilde{\beta}_{k,M}^{-1} \epsilon_{\overline{N}} f_{\text{out}} \sigma_{\overline{M}} \hat{\beta}_{k,M} \\ 0 & \tilde{\beta}_{k,N}^{-1} \pi f_{\text{out}} \iota_{\overline{M}} \tilde{\beta}_{k,M} & \tilde{\beta}_{k,N}^{-1} \pi f_{\text{out}} \sigma_{\overline{M}} \hat{\beta}_{k,M} \end{pmatrix},$$

remember that  $f'_i = f_i$  if  $i \neq k$ .

To finish the proof we need to give an natural isomorphism  $\psi_{k,M} : M'_k \rightarrow M_k$  such that

$$(5.9) \quad \psi_{k,M} \alpha_{k,M'} = \alpha_{k,M}, \quad \beta_{k,M} \psi_{k,M} = \beta_{k,M'}.$$

If  $i \neq k$  we set  $\psi_{i,M} = \text{id}_{M_i}$ , and we denote  $\psi_M = (\psi_{i,M})_{i \in Q_0}$ . From the expression (5.6) we observe that have the following filtration

$$(5.10) \quad \{0\} \subseteq \ker(\beta_{k,M}) \subseteq \text{im}(\alpha_{k,M}) \subseteq M_k$$

It can be checked that we can choose sections

$$(5.11) \quad \begin{aligned} \sigma_{1,M} &: \text{im}(\alpha_{k,M}) / \ker(\beta_{k,M}) \longrightarrow \text{im}(\alpha_{k,M}), \\ \sigma_{2,M} &: M_k / \text{im}(\alpha_{k,M}) \longrightarrow M_k; \end{aligned}$$

with fulfill

$$(5.12) \quad \text{im}(\sigma_{1,M}) = \alpha_{k,M}(\ker(\bar{\rho})), \quad \text{im}(\beta_{k,M}\sigma_{2,M}) = \text{im}(\bar{\sigma})$$

If we define  $\psi_{k,M} = (-\iota, -\iota\sigma_{1,M}, -\iota\sigma_{2,M})$ , then it can be proved that  $\psi$  is a isomorphism. By multiplying the respective matrix and taking into account (5.11) we get (5.9).

Since  $f\psi_M - \psi_N\mu_k^-\mu_k^+(f)$  is confined to  $k$ , we have the commutative diagram in  $\Lambda \text{ mod } [\text{add } S_k]$ .

$$(5.13) \quad \begin{array}{ccc} \mu_k^-\mu_k^+(M) & \xrightarrow{\psi_M} & M \\ \mu_k^-\mu_k^+(f) \downarrow & & \downarrow f \\ \mu_k^-\mu_k^+(N) & \xrightarrow{\psi_N} & N \end{array}$$

The other composition  $\mu_k^+\mu_k^-$  is similar. Therefore the result follows.  $\square$

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